

DIAMETER CONTROLS AND SMOOTH CONVERGENCE AWAY FROM SINGULAR SETS

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ABSTRACT. We prove that if a family of metrics, g_i , on a compact Riemannian manifold, M^n , have a uniform lower Ricci curvature bound and converge to g_∞ smoothly away from a singular set, S , with Hausdorff measure, $H^{n-1}(S) = 0$, and if there exists connected precompact exhaustion, W_j , of $M^n \setminus S$ satisfying $\text{diam}_{g_i}(M^n) \leq D_0$, $\text{Vol}_{g_i}(\partial W_j) \leq A_0$ and $\text{Vol}_{g_i}(M^n \setminus W_j) \leq V_j$ where $\lim_{j \rightarrow \infty} V_j = 0$ then the Gromov-Hausdorff limit exists and agrees with the metric completion of $(M^n \setminus S, g_\infty)$. Recall that in the prior work with Sormani the same conclusion is reached but the singular set is assumed to be a submanifold of codimension two. We have a second main theorem in which the Hausdorff measure condition on S is replaced by diameter estimates on the connected components of the boundary of the exhaustion, ∂W_j . In addition, we show that the uniform lower Ricci curvature bounds in these theorems can be replaced by the existence of a uniform linear contractibility function. If this condition is removed altogether, then we prove that $\lim_{j \rightarrow \infty} d_{\mathcal{F}}(M'_j, N') = 0$, in which M'_j and N' are the settled completions of (M, g_j) and $(M_\infty \setminus S, g_\infty)$ respectively and $d_{\mathcal{F}}$ is the Sormani-Wenger Intrinsic Flat distance. We present examples demonstrating the necessity of many of the hypotheses in our theorems.

1. INTRODUCTION

In this paper, we will prove results concerning the smooth convergence of Riemannian metrics away from a singular set S . One definition of smooth convergence away from singularities is as follows:

Definition 1.1. *We will say that a sequence of Riemannian metrics g_i on a compact manifold M^n converges smoothly away from $S \subset M^n$ to a Riemannian metric g_∞ on $M^n \setminus S$ if for every compact set $K \subset M^n \setminus S$, g_i converge $C^{k,\alpha}$ smoothly to g_∞ as tensors.*

Right away from the definition, it is apparent that the global geometry is not well controlled under such convergence. It is natural to ask under what additional conditions the original sequence of manifolds, $M_i = (M^n, g_i)$ have the expected Gromov-Hausdorff (GH) and Sormani-Wenger Intrinsic Flat (SWIF) limits [Gro99] [SW11]. Recall that there are examples of sequences of metrics on spheres which converge smoothly away from a point singularity which have no subsequence converging in the GH or the SWIF sense, so additional conditions are necessary (c.f. [LS]).

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Many results concerning GH limits of the M_i have appeared in the literature. For example, Anderson in [And89] studies the convergence of Einstein metrics to orbifolds. Bando-Kasue-Nakajima in [BKN89] studies the singularities of the Einstein ALF manifolds. Eyssidieux-Guedj-Zeriahi in [EGZ09] prove similar results for the solutions to the complex Monge-Ampere equation. Also Huang in [Hua09], Ruan-Zhong in [RZ], Sesum in [Ses04], Tian in [Tia90] and Tosatti in [Tos09] study the convergence of Kahler-Einstein metrics and Kahler-Einstein orbifolds. However, even in this setting, the relationship is not completely clear and the limits need not agree (see [Ban90].) In Tian-Viaclovsky [TV05], compactness results for various classes Riemannian metrics in dimension four were obtained in particular for anti-self-dual metrics, Kahler metrics with constant scalar curvature, and metrics with harmonic curvature. Also the relation between different notions of convergence for Ricci flow is studied in the upcoming paper of the author [Lak].

In the prior work of the author with Sormani [LS], new results on the GH limits of the M_i were obtained by first studying their SWIF limits and then applying various means of relating SWIF and GH limits. In the most general theorem in that paper, we assume the existence of a connected precompact exhaustion, W_j of M , with diameter, volume and area bounds as in our Theorem 1.2 as well as an additional condition of “uniform well embeddedness” and then concludes that the SWIF limit exists and is the settled completion of $(M \setminus S, g_\infty)$. These notions are reviewed and this theorem is stated precisely in Section 2 (see Definitions 2.2 and 2.9 and Theorem 2.10). Here we build upon that theorem by providing easily tested hypothesis that imply this uniformly well embedded criteria to prove results about both the SWIF and GH limits of the Riemannian manifolds. There we proved that uniform embeddedness is obtained whenever the singular set is a submanifold of codimension 2, and here we prove the singular set need not have any regularity:

Theorem 1.2. *Let $M_i = (M^n, g_i)$ be a sequence of compact oriented Riemannian manifolds such that there is a subset, S , with $H^{n-1}(S) = 0$ and connected precompact exhaustion, W_j , of $M \setminus S$ satisfying (8) with g_i converge smoothly to g_∞ on each W_j ,*

$$(1) \quad \text{diam}_{M_i}(W_j) \leq D_0 \quad \forall i \geq j,$$

$$(2) \quad \text{Vol}_{g_i}(\partial W_j) \leq A_0,$$

and

$$(3) \quad \text{Vol}_{g_i}(M \setminus W_j) \leq V_j \text{ where } \lim_{j \rightarrow \infty} V_j = 0.$$

Then

$$(4) \quad \lim_{i \rightarrow \infty} d_{\mathcal{F}}(M'_i, N') = 0.$$

where M'_i and N' are the settled completion of (M, g_i) and $(M \setminus S, g_\infty)$ respectively.

Here $\text{diam}_M(W)$ is the extrinsic diameter found by

$$(5) \quad \text{diam}_M(W) = \sup\{d_M(x, y) : x, y \in W\}$$

where d_M is the extrinsic distance measured in M rather than W :

$$(6) \quad d_M(x, y) = \inf\{L(C) : C : [0, 1] \rightarrow M, C(0) = x, C(1) = y\}.$$

We write $M_i = (M, g_i)$. The intrinsic diameter of W is then $\text{diam}_W(W)$. See Remark 4.3 for the necessity of the hypotheses in Theorem 1.2.

Adding a Ricci curvature condition and applying methods of [LS] we obtain a result of the Gromov-Hausdorff limit of the sequence:

Theorem 1.3. *Let $M_i = (M, g_i)$ be a sequence of oriented compact Riemannian manifolds with uniform lower Ricci curvature bounds,*

$$(7) \quad \text{Ricci}_{g_i}(V, V) \geq (n-1)H g_i(V, V) \quad \forall V \in TM_i,$$

which converges smoothly away from a singular set, S , with $H^{n-1}(S) = 0$. If there is a connected precompact exhaustion, W_j , of $M \setminus S$,

$$(8) \quad \bar{W}_j \subset W_{j+1} \text{ with } \bigcup_{j=1}^{\infty} W_j = M \setminus S,$$

satisfying

$$(9) \quad \text{diam}(M_i) \leq D_0,$$

$$(10) \quad \text{Vol}_{g_i}(\partial W_j) \leq A_0,$$

and

$$(11) \quad \text{Vol}_{g_i}(M \setminus W_j) \leq V_j \text{ where } \lim_{j \rightarrow \infty} V_j = 0,$$

then

$$(12) \quad \lim_{j \rightarrow \infty} d_{GH}(M_j, N) = 0,$$

where N is the metric completion of $(M \setminus S, g_{\infty})$.

See Remark 4.3 for the necessity of our hypotheses in Theorem 1.3. We may replace the Ricci condition by a condition on contractibility (see Theorem 4.6). For the necessity of the hypotheses in this theorem see [LS, Remark 6.8].

We also prove that the condition on the Hausdorff measure of the singular set can be replaced by imposing diameter bounds on the exhaustion's boundaries:

Theorem 1.4. *Let $M_i = (M, g_i)$ be a sequence of Riemannian manifolds such that there is a closed subset, S , and a connected precompact exhaustion, W_j , of $M \setminus S$ satisfying (8) such that g_i converge smoothly to g_{∞} on each W_j .*

If each connected component of $M \setminus W_j$ has a connected boundary,

$$(13) \quad \limsup_{i \rightarrow \infty} \left\{ \sum_{\beta} \text{diam}_{(\Omega_j^{\beta}, g_i)}(\Omega_j^{\beta}) : \Omega_j^{\beta} \text{ connected component of } \partial W_j \right\} \leq B_j,$$

where $\lim_{j \rightarrow \infty} B_j = 0$, and if we have

$$(14) \quad \text{diam}_{(W_j, g_i)}(W_j) \leq D_{int},$$

$$(15) \quad \text{Vol}_{g_i}(\partial W_j) \leq A_0,$$

$$(16) \quad \text{Vol}_{g_i}(M \setminus W_j) \leq V_j \text{ where } \lim_{j \rightarrow \infty} V_j = 0,$$

then

$$(17) \quad \lim_{j \rightarrow \infty} d_{\mathcal{F}}(M'_j, N') = 0.$$

where N' is the settled completion of $(M \setminus S, g_\infty)$.

See Remark 5.2 for the necessity of our hypotheses in Theorem 1.4.

In presence of a uniform lower Ricci curvature bound, Theorem 1.4 can be applied to prove the following theorem:

Theorem 1.5. *Let $M_i = (M, g_i)$ be a sequence of oriented Riemannian manifolds with uniform lower Ricci curvature bounds,*

$$(18) \quad \text{Ricci}_{g_i}(V, V) \geq (n-1)H g_i(V, V) \quad \forall V \in TM_i,$$

which converges smoothly away from a closed singular set, S .

If there is a connected precompact exhaustion, W_j , of $M \setminus S$, satisfying (8) such that each connected component of $M \setminus W_j$ has a connected boundary,

$$(19) \quad \limsup_{i \rightarrow \infty} \left\{ \sum_{\beta} \text{diam}_{(\Omega_j^\beta, g_i)}(\Omega_j^\beta) : \Omega_j^\beta \text{ connected component of } \partial W_j \right\} \leq B_j,$$

where $\lim_{j \rightarrow \infty} B_j = 0$, and if we have

$$(20) \quad \text{diam}(M_i) \leq D_0,$$

$$(21) \quad \text{diam}_{(W_j, g_i)}(W_j) \leq D_{\text{int}},$$

and volume controls:

$$(22) \quad \text{Vol}_{g_i}(\partial W_j) \leq A_0,$$

$$(23) \quad \text{Vol}_{g_i}(M \setminus W_j) \leq V_j \text{ such that } \lim_{j \rightarrow \infty} V_j = 0,$$

then

$$(24) \quad \lim_{j \rightarrow \infty} d_{GH}(M_j, N) = 0.$$

where N is the metric completion of $(M \setminus S, g_\infty)$.

In Theorems 1.3 and 1.5, the diameter hypothesis $\text{diam}(M_i) \leq D_0$ is not necessary when the Ricci curvature is nonnegative (see Lemma 2.14.)

The Ricci curvature condition in Theorems 1.3 and 1.5 may be replaced by a requirement that the sequence of manifolds have a uniform linear contractibility function (see Theorem 5.5 and Theorem 4.6.) See Definition 2.4 for the definition of a contractibility function. Recall that Greene-Petersen have a compactness theorem for sequences of manifolds with uniform contractibility functions and upper bounds on their volume [GPV92].

The paper is organized as follows. In Section 2, we will briefly review all the notions and theorems that we have used in this paper. Section 3 discusses a few interesting examples to illustrate the underlying phenomena and also to prove the necessity of some of our hypotheses. In Section 4, we give a proof of our theorems which assume $\mathcal{H}^{n-1}(S) = 0$ [Theorem 1.2, Theorem 1.3 and Theorem 4.6]. Section 5 is devoted to the proof of our theorems which replace the Hausdorff measure hypothesis with diameter bounds [Theorem 1.4, Theorem 1.5 and Theorem 5.5].

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2. BACKGROUND

2.1. Metric Completion and Settled Completion. We give a very brief review of the definitions of metric completion and settled completion of a metric space:

Definition 2.1. *Given a precompact metric space, X , the metric completion, \bar{X} of X is the space of Cauchy sequences, $\{x_j\}$, in X with the metric*

$$(25) \quad d(\{x_j\}, \{y_j\}) = \lim_{j \rightarrow \infty} d_X(x_j, y_j),$$

and where two Cauchy sequences are identified if the distance between them is 0. There is an isometric embedding, $\varphi : X \rightarrow \bar{X}$, defined by $\varphi(x) = \{x\}$ where $\{x\}$ is a constant sequence. Lipschitz functions, $F : X \rightarrow Y$, extend to $F : \bar{X} \rightarrow Y$ via $F(\{x_j\}) = \lim_{j \rightarrow \infty} F(x_j)$ as long as Y is complete.

The definition of the settled completion which is essential in studying the Intrinsic Flat convergence of metric spaces is as follows

Definition 2.2 (Sormani-Wenger [SW11]). *The settled completion, X' , of a metric space X with a measure μ is the collection of points x in the metric completion \bar{X} which have positive lower density*

$$(26) \quad \liminf_{r \rightarrow 0} \mu(B_p(r))/r^m > 0.$$

The resulting space is then “completely settled”.

2.2. GH and SWIF Distances. The Gromov-Hausdorff (GH) distance was defined by Gromov as

$$(27) \quad d_{GH}(X_1, X_2) = \inf \left\{ d_H^Z(\varphi_1(X_1), \varphi_2(X_2)) : \varphi_i : X_i \rightarrow Z \right\}$$

where the infimum is taken over all common metric spaces, Z , and all isometric embeddings, $\varphi_i : X_i \rightarrow Z$. Limit spaces obtained from the Gromov-Hausdorff convergence are compact metric spaces [Gro99].

The Sormani Wenger Intrinsic Flat (SWIF) distance is defined similarly by replacing the Hausdorff distance by the flat distance, viewing $\varphi_i(M_i)$ as integral current spaces in the sense of [AK00]. The limit spaces obtained under intrinsic flat convergence are “integral current spaces”: completely settled metric spaces with an integral current structure that defines a notion of integration over m -forms. When the limit is the settled completion of an open manifold, this the integral current structure is simply defined by integration over that open manifold (see [SW11] for more details.)

The SWIF distance, $d_{\mathcal{F}}(M_1, M_2)$, is estimated by explicitly constructing a filling manifold, B^{m+1} , between the two given manifolds, finding the excess boundary manifold A^m satisfying

$$(28) \quad \int_{\varphi_1(M_1)} \omega - \int_{\varphi_2(M_2)} \omega = \int_B d\omega + \int_A \omega,$$

and summing their volumes

$$(29) \quad d_{\mathcal{F}}(M_1^m, M_2^m) \leq \text{Vol}_m(A^m) + \text{Vol}_{m+1}(B^{m+1}).$$

In the next subsection we present review a theorem which clarifies the concept of the intrinsic flat distance while proving a means of estimating it.

2.3. Estimating the Gromov Hausdorff and Intrinsic Flat Distance. We can estimate both of these distances by applying the following theorem which was proven in prior work of the author with Sormani [LS] by constructing an explicit space Z and isometric embeddings φ_i . Here we have cut and pasted the exact theorem statement along with the corresponding figure from that paper:

Theorem 2.3. *Suppose $M_1 = (M, g_1)$ and $M_2 = (M, g_2)$ are oriented precompact Riemannian manifolds with diffeomorphic subregions $U_i \subset M_i$ and diffeomorphisms $\psi_i : U \rightarrow U_i$ such that*

$$(30) \quad \psi_1^* g_1(V, V) < (1 + \epsilon)^2 \psi_2^* g_2(V, V) \quad \forall V \in TU,$$

and

$$(31) \quad \psi_2^* g_2(V, V) < (1 + \epsilon)^2 \psi_1^* g_1(V, V) \quad \forall V \in TU.$$

Taking the extrinsic diameters,

$$(32) \quad D_{U_i} = \sup\{\text{diam}_{M_i}(W) : W \text{ is a connected component of } U_i\} \leq \text{diam}(M_i).$$

we define a hemispherical width,

$$(33) \quad a > \frac{\arccos(1 + \epsilon)^{-1}}{\pi} \max\{D_{U_1}, D_{U_2}\}.$$

Taking the difference in distances with respect to the outside manifolds,

$$(34) \quad \lambda = \sup_{x, y \in U} |d_{M_1}(\psi_1(x), \psi_1(y)) - d_{M_2}(\psi_2(x), \psi_2(y))|,$$

we define heights,

$$(35) \quad h = \sqrt{\lambda(\max\{D_{U_1}, D_{U_2}\} + \lambda/4)},$$

and

$$(36) \quad \bar{h} = \max\{h, \sqrt{\epsilon^2 + 2\epsilon} D_{U_1}, \sqrt{\epsilon^2 + 2\epsilon} D_{U_2}\}.$$

Then the Gromov-Hausdorff distance *between the metric completions* is bounded,

$$(37) \quad d_{GH}(\bar{M}_1, \bar{M}_2) \leq a + 2\bar{h} + \max\{d_H^{M_1}(U_1, M_1), d_H^{M_2}(U_2, M_2)\},$$

and the intrinsic flat distance *between the settled completions* is bounded,

$$\begin{aligned} d_{\mathcal{F}}(M'_1, M'_2) \leq & (2\bar{h} + a) \left(\text{Vol}_m(U_1) + \text{Vol}_m(U_2) + \text{Vol}_{m-1}(\partial U_1) + \text{Vol}_{m-1}(\partial U_2) \right) \\ & + \text{Vol}_m(M_1 \setminus U_1) + \text{Vol}_m(M_2 \setminus U_2), \end{aligned}$$

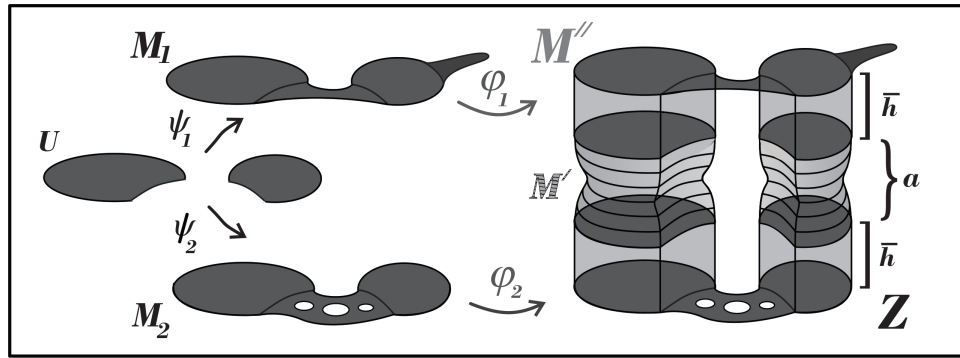


FIGURE 1. Creating Z for Theorem 2.3.

Note that permission to reprint this figure along with the statement of Theorem 2.3 has been granted by the author and Christina Sormani who own the copyright to this figure that first appeared in [LS].

2.4. Review of Compactness Theorems.

Definition 2.4. A function $\rho : [0, r_0] \rightarrow [0, \infty)$ is a contractibility function for a manifold M with metric g if every ball $B_p(r)$ is contractible within $B_p(\rho(r))$.

Theorem 2.5 (Gromov [Gro99]). A sequence of compact Riemannian manifolds, (M_j, g_j) , such that $\text{diam}(M_j) \leq D$ and $\text{Ricci}_{M_j} \geq -H$, has a subsequence converging in the Gromov-Hausdorff sense to a metric space (X, d) .

Theorem 2.6 (Greene-Petersen[GPV92]). A sequence of compact Riemannian manifolds, (M_j, g_j) , such that $\text{Vol}(M_j) \leq V$ and such that there is a uniform contractibility function, $\rho : [0, r_0] \rightarrow [0, \infty)$, for all the M_j , has a subsequence converging in the Gromov-Hausdorff sense to a metric space (X, d) .

In [SW10] the following theorems were proven which can be applied to deduce information about the Gromov-Hausdorff limit of a sequence.

Theorem 2.7 (Sormani-Wenger [SW11]). If a sequence of oriented compact Riemannian manifolds, (M_j, g_j) , with a uniform linear contractibility function, $\rho : [0, \infty) \rightarrow [0, \infty)$ and a uniform upper bound on volume, $\text{Vol}(M_j) \leq V$, converges

in the Gromov-Hausdorff sense to (X, d) , then it converges in the intrinsic flat Sense to (X, d, T) (see Theorem 4.14 of [SW11]).

Recall that, in general, the intrinsic flat limits and Gromov-Hausdorff limits need not agree [LS, Examples 2.3 and 2.3] because intrinsic flat limits do not include points with 0 density as in (26). In fact intrinsic flat limits may exist when Gromov-Hausdorff limits do not [Example 3.11].

Theorem 2.8 (Sormani-Wenger). *If a sequence of oriented compact Riemannian manifolds, (M_j, g_j) , such that $\text{diam}(M_j) \leq D$ and $\text{Ricci}_{M_j} \geq 0$ and $\text{vol}(M_j) \geq V_0$ converges in the Gromov-Hausdorff sense to (X, d) , then it converges in the intrinsic flat Sense to (X, d, T) (see Theorem 4.16 of [SW11]).*

This theorem is conjectured to hold with uniform lower bounds on Ricci curvature [SW11].

2.5. Review of Smooth Convergence away from Singular Sets. Here we review results from our prior work with Sormani in [LS]. We need the following key definition (a bound on the metric distortions) in order to state the results. Recall the definition of the extrinsic distance in (6).

Definition 2.9. *Given a sequence of Riemannian manifolds $M_i = (M, g_i)$ and an open subset, $U \subset M$, a connected precompact exhaustion, W_j , of U satisfying (8) is **uniformly well embedded** if there exist a λ_0 such that*

$$(38) \quad \limsup_{j \rightarrow \infty} \limsup_{k \rightarrow \infty} \limsup_{i \rightarrow \infty} \lambda_{i,j,k} \leq \lambda_0,$$

and

$$(39) \quad \limsup_{k \rightarrow \infty} \lambda_{i,j,k} = \lambda_{i,j} \text{ where } \limsup_{i \rightarrow \infty} \lambda_{i,j} = \lambda_j \text{ and } \lim_{j \rightarrow \infty} \lambda_j = 0.$$

where,

$$(40) \quad \lambda_{i,j,k} = \sup_{x,y \in W_j} |d_{(W_k, g_i)}(x, y) - d_{(M, g_i)}(x, y)|.$$

The author and Sormani in [LS] have proven:

Theorem 2.10. *Let $M_i = (M, g_i)$ be a sequence of Riemannian manifolds such that there is a closed subset, S , and a uniformly well embedded connected precompact exhaustion, W_j , of $M \setminus S$ satisfying (8) such that g_i converge smoothly to g_∞ on each W_j with*

$$(41) \quad \text{diam}_{M_i}(W_j) \leq D_0 \quad \forall i \geq j,$$

$$(42) \quad \text{Vol}_{g_i}(\partial W_j) \leq A_0,$$

and

$$(43) \quad \text{Vol}_{g_i}(M \setminus W_j) \leq V_j \text{ where } \lim_{j \rightarrow \infty} V_j = 0,$$

Then

$$(44) \quad \lim_{j \rightarrow \infty} d_{\mathcal{F}}(M'_j, N') = 0.$$

where N' is the settled completion of $(M \setminus S, g_\infty)$.

Remark 2.11. Example 3.6 demonstrates the necessity of well-embeddedness condition in Theorem 2.10.

Lemma 2.12. Let $M_i = (M, g_i)$ be a sequence of Riemannian manifolds such that there is a closed subset, S , and a connected precompact exhaustion, W_j , of $M \setminus S$ satisfying (8) such that g_i converge smoothly to g_∞ on each W_j . If $\text{Vol}_{g_\infty}(M \setminus S) < \infty$ and

$$(45) \quad \text{Vol}_{g_i}(M \setminus W_j) \leq V_j \text{ where } \lim_{j \rightarrow \infty} V_j = 0,$$

then there exists a uniform $V_0 > 0$ such that

$$(46) \quad \text{Vol}_{g_i}(M) < V_0.$$

Proposition 2.13. Suppose we have a sequence of manifolds, $M_j = (M, g_j)$ with a uniform lower bound on Ricci curvature and

$$(47) \quad \text{Vol}(M_j) \leq V_0,$$

converging smoothly away from a singular set to $(M \setminus S, g_\infty)$. Suppose also that (M, g_j) converge in the intrinsic flat sense to N' where N' is the settled completion of $(M \setminus S, g_\infty)$. Then

$$(48) \quad d_{GH}(\bar{M}_j, \bar{N}) \rightarrow 0,$$

and $\bar{N} = N'$.

Lemma 2.14. Suppose we have a sequence of manifolds, $M_j = (M, g_j)$ with non-negative Ricci curvature and

$$(49) \quad \text{Vol}(M_j) \leq V_0,$$

converging smoothly away from a singular set to $(M \setminus S, g_\infty)$ then

$$(50) \quad \text{diam}_{M_i}(W_j) \leq \text{diam}(M_i) \leq D_0 \quad \forall i \geq j.$$

Theorem 2.15. Let $M_i = (M, g_i)$ be a sequence of compact oriented Riemannian manifolds with a uniform linear contractibility function, ρ , which converges smoothly away from a singular set, S . If there is a uniformly well embedded connected precompact exhaustion of $M \setminus S$ as in (8) satisfying the volume conditions (195) and (196) then

$$(51) \quad \lim_{j \rightarrow \infty} d_{GH}(M_j, N) = 0,$$

where N is the settled and metric completion of $(M \setminus S, g_\infty)$.

Proof. By Lemma 2.12, we have

$$(52) \quad \text{Vol}(M_i) \leq V_0.$$

This combined with the uniform contractibility function allows us to apply the Greene-Petersen Compactness Theorem. In particular we have a uniform upper bound on diameter:

$$(53) \quad \text{diam}(M_i) \leq D_0,$$

We may now apply Theorem 2.10 to obtain

$$(54) \quad \lim_{j \rightarrow \infty} d_{\mathcal{F}}(M_j, N') = 0$$

We then apply Theorem 2.7 to see that the flat limit and Gromov-Hausdorff limits agree due to the existence of the uniform linear contractibility function and the fact that the volume is bounded below uniformly by the smooth limit. In particular the metric completion and the settled completion agree. \square

3. EXAMPLES

In this section we present some examples which helps in understanding the notions we have mentioned so far. Some examples will prove the necessity of some conditions in Theorem 1.5.

3.1. Unbounded Limits. The following examples show why some sort of bounded geometry is necessary in this context.

Example 3.1. *There are metrics g_j on the sphere M^3 with a uniform upper bound on volume such that (M^3, g_j) converge smoothly away from a point singularity $S = \{p_0\}$ to a complete noncompact manifold. There is no Gromov-Hausdorff limit in this case. The intrinsic flat limit is $(M \setminus S, g_\infty)$.*

Proof. Let

$$(55) \quad g_0 = h^2(r)dr^2 + f^2(r)g_{S^2},$$

be defined on $M^3 \setminus S$ as a complete metric such that

$$(56) \quad \int_0^\pi h(r)dr = \infty,$$

and

$$(57) \quad \int_0^\pi \omega_2 h(r) f^2(r) dr < \infty,$$

so that $\text{diam}(M \setminus S, g_0) = \infty$ and $\text{Vol}(M \setminus S, g_0) < \infty$.

We set

$$(58) \quad g_j = h_j^2(r)dr^2 + f_j^2(r)g_{S^2},$$

such that

$$(59) \quad h_j(r) = h(r) \quad r \in [0, \pi - 1/j],$$

$$(60) \quad f_j(r) = f(r) \quad r \in [0, \pi - 1/j],$$

and extend smoothly so that g_j is a metric on S^3 .

Metrics g_j converge smoothly to g_0 away from $S = \{p_0\} = r^{-1}(\pi)$ and, since $(M \setminus S, g_0)$ is noncompact, (M, g_j) has no Gromov-Hausdorff limit. The intrinsic

flat limit of (M, g_j) is the settled completion of $(M \setminus S, g_0)$ by Theorem 1.2, taking $W_j = r^{-1}[0, \pi - 1/j)$ since

$$(61) \quad \int_{\pi-1/k}^{\pi} \omega_2 h(r) f^2(r) dr = 0,$$

by the finiteness of (57) and we also have $\text{Vol}_{g_i}(\partial(W_j)) \leq f^2(r)$. In this case the settled completion is just $(M \setminus S, g_0)$ because it is already a complete metric space with positive density. \square

Example 3.2. *There are metrics g_j on $M^3 = S^3$ converging smoothly away from a singular set $S = \{p_0\}$ to a complete noncompact manifold of infinite volume. (M, g_j) have no intrinsic flat or Gromov-Hausdorff limit since, if such a limit existed it would have to contain the smooth limit and the smooth limit has infinite diameter and volume.*

Proof. We define a metric g_0 on $M \setminus S$ exactly as in Example 3.1 except that we replace (57) with

$$(62) \quad \int_0^{\pi} \omega_2 h(r) f^2(r) dr = \infty,$$

so that $\text{diam}(M \setminus S, g_0) = \infty$ and $\text{Vol}(M \setminus S, g_0) = \infty$.

Selecting g_j also as in that example, we have (M, g_j) converge smoothly away from S to $(M \setminus S, g_0)$. However there is no Gromov-Hausdorff limit because the diameter diverges to infinity [Gro99] and there is no intrinsic flat limit because the volume diverges to infinity [SW11]. \square

One may define pointed Gromov-Hausdorff and pointed intrinsic flat limits to deal with unboundedness. However even assuming boundedness, we see in [LS, Example 3.11] that the Gromov-Hausdorff limit need not exist.

3.2. Ricci Example. This example shows that the mere uniform lower bound for Ricci curvature does not imply the existence of the Gromov-Hausdorff limit.

Example 3.3. *There are metrics g_j on $M^3 = S^3$ with negative uniform lower bound on Ricci curvature, converging smoothly away from a singular set $S = \{p_0\}$ to a complete noncompact manifold of finite volume.*

Proof. Consider the metric g_0 on $S^3 \setminus \{p_0\} = \mathbb{R} \times S^2$ given by

$$(63) \quad \bar{g}(t) = dt^2 + (\bar{f}(t))^2 g_{S^2},$$

where, \bar{f} is a nonzero smooth function such that

$$(64) \quad \bar{f}(t) = \sin(t) \text{ for } t \in [0, \pi/2],$$

$$(65) \quad \bar{f}(t) = \exp -t \text{ for } t \in [\pi/2 + 1, \infty),$$

with $\bar{f}''(t) < \bar{f}(t)$ elsewhere. hence, \bar{g} has Ricci curvature bounded below by

$$(66) \quad -\Lambda = -2 \max \frac{\bar{f}''}{\bar{f}} > -\infty.$$

We can extract warped metrics \bar{g}_j on $[0, j+1] \times S^2$

$$(67) \quad \bar{g}(t) = dt^2 + \bar{f}_j(t)^2 g_{S^2},$$

where \bar{f}_j is a nonzero smooth function satisfying

$$(68) \quad \bar{f}_j(t) = \bar{f}(t) \text{ for } t \in [0, j-1],$$

$$(69) \quad \bar{f}_j(t) = \exp -j \sin(\exp j(\pi + t - j - 1)) \text{ for } t \in [j, j+1],$$

and

$$(70) \quad -2 \max \frac{\bar{f}_j''}{\bar{f}_j} \geq -\Lambda.$$

Note that in fact we are cutting off a part of \bar{f} and replacing it with a less concave function which closes up like a sin function hence obtaining a metric on S^3 , with lower bound on Ricci curvature.

It is clear that

$$(71) \quad \text{Vol}(M, g_0) \leq \infty.$$

Let $\phi : [0, \pi] \rightarrow [0, \infty)$ be a smooth increasing function such that

$$(72) \quad \phi(r) = r \text{ for } r \in [0, \pi/2],$$

with

$$(73) \quad \lim_{r \rightarrow \pi} \phi(r) = \infty.$$

For $j > 2$, let $\phi_j(r) : [0, \pi] \rightarrow [0, L_j = j + \pi/2 + 1]$ be a smooth increasing function such that

$$(74) \quad \phi_j(r) = \phi(r) \text{ for } r \in [0, \phi^{-1}(j + \pi/2)],$$

and

$$(75) \quad \phi_j(r) = j + r - \pi/2 + 1 \text{ for } r \text{ near } \pi.$$

we construct metrics

$$(76) \quad g_j(r) = \phi_j^*(\bar{g}_j),$$

with Ricci bounded below by $-\Lambda$ converging smoothly away from $\{p_0\}$ to $\phi^*(\bar{g})$. Taking $W_j = r^{-1}([0, \pi - 1/j])$ we observe that W_j satisfies all the hypotheses in Theorem 1.5 except that $\text{diam}_{M_i}(W_j)$ is not bounded. The Gromov-Hausdorff limit does not exist because $(M \setminus S, g_0)$ is complete noncompact. Both intrinsic flat limit and the metric completion coincide with the complete noncompact manifold $(M \setminus S, g_0)$. \square

Remark 3.4. From [Yau76], we know that any complete noncompact manifold with nonnegative Ricci curvature has infinite volume, so Example 3.3 can not be constructed with the sequence having nonnegative Ricci curvature.

3.3. Pinching a torus.

Example 3.5. *There are (M^2, g_j) all diffeomorphic to the torus, $S^1 \times S^1$ which converge smoothly away from a singular set, $S = \{0\} \times S^1$, to*

$$(77) \quad (M \setminus S, g_\infty) = \left((0, 2\pi) \times S^1, dt^2 + \sin^2\left(\frac{t}{2}\right) ds^2 \right).$$

So the metric completion and the settled completions are both homeomorphic to

$$(78) \quad M_\infty = [0, 2\pi] \times S^1 / \sim,$$

where

$$(79) \quad (0, s_1) \sim (0, s_2) \text{ and } (2\pi, s_1) \sim (2\pi, s_2) \quad \forall s_1, s_2 \in S^1.$$

However the Gromov-Hausdorff and Intrinsic Flat limits identify these two end points.

Proof. Let g_j on M be defined by

$$(80) \quad g_j = dt^2 + f_j^2(t) ds^2,$$

where $f_j : S^1 \rightarrow (0, 1]$ are smooth with $|f_j'(t)| \leq 1$ that decrease uniformly to $\sin(\frac{t}{2})$ and $f_j(t) = \sin(\frac{t}{2})$ for $t \in [1/j, 2\pi - 1/j]$. \square

3.4. Examples of Slit Tori.

Example 3.6. *Let (M^2, g) be the standard flat 2 torus $S^1 \times S^1$ and $S \subset M^2$ a vertical geodesic segment of length $\leq \pi$, then if g_j are a constant sequence of the standard flat metric, we see that (M^2, g_j) converges smoothly to itself and thus the intrinsic flat and Gromov-Hausdorff limits are both the flat torus. However, the metric completion of $(M \setminus S, g_\infty)$ has two copies of the slit, S (with end points identified hence the limit has fundamental group $= \mathbb{Z}$) one found taking limits of Cauchy sequences from the right and the other found taking limits of Cauchy sequences from the left. This example shows necessity of uniform well embeddedness condition in our Theorems.*

Proof. Let $M^2 = S^1 \times S^1 = [0, 2\pi] \times [0, 2\pi] / \sim$ such that $(x, 0) \sim (x, 2\pi)$ and $(0, y) \sim (2\pi, y)$. Without loss of generality, we can assume $S = \{(\pi, y) : y \in [\pi/2, 3\pi/2]\}$. Then the metric completion of $M^2 \setminus S$ is

$$(81) \quad M_\infty = \frac{S^1 \times S^1 \times \{0\} \sqcup S^1 \times S^1 \times \{1\}}{\sim},$$

where,

$$(82) \quad (x, y, 0) \sim (x, y, 1) \text{ for } (x, y) \notin S,$$

with the distance d_∞ given by

$$(83) \quad d_\infty([x, y, l], [x', y', l']) = \lim_{\delta \rightarrow 0^+} d_{M^2 \setminus S}((x + (-1)^l \delta, y), (x' + (-1)^{l'} \delta, y'))$$

for $l, l' = 0, 1$. In particular,

$$(84) \quad d_\infty([\pi, \pi, 0], [\pi, \pi, 1]) = \pi.$$

Notice that M_∞ is not a manifold (not even Hausdorff as $B_r([\pi, \pi, 0]) \cap B_r([\pi, \pi, 1]) \neq \emptyset$ for all r, r' .) Taking the connected precompact exhaustion

$$(85) \quad W_j = M^2 \setminus ([\pi/2 - 1/j, 3\pi/2 + 1/j] \times [\pi - 1/j, \pi + 1/j]),$$

we observe that

$$(86) \quad \begin{aligned} \text{diam}_{M_i}(W_j) &\leq \text{diam}(M^2) \\ \text{Vol}(M_i) &= \text{Vol}(M^2) \\ \text{Vol}_{g_i}(\partial W_j) &\leq 2\pi + 4, \end{aligned}$$

are uniformly bounded, also

$$(87) \quad \lim_{j \rightarrow \infty} \text{Vol}_{g_i}(N \setminus W_j) = \lim_{j \rightarrow \infty} (2/j)(\pi + 2/j) = 0,$$

but,

$$(88) \quad \begin{aligned} \lambda_{i,j,k} &= \sup_{x,y \in W_j} |d_{(W_k, g_i)}(x, y) - d_{(M, g_i)}(x, y)| \\ &\geq |d_{(W_k, g_i)}((\pi - 1/j, \pi), (\pi + 1/j, \pi)) - d_{(M, g_i)}((\pi - 1/j, \pi), (\pi + 1/j, \pi))| \\ &\geq \pi + 2/j + 2/k, \end{aligned}$$

Therefore,

$$(89) \quad \lim_{j \rightarrow \infty} \limsup_{i \rightarrow \infty} \limsup_{k \rightarrow \infty} \lambda_{i,j,k} \geq \lim_{j \rightarrow \infty} \limsup_{i \rightarrow \infty} \limsup_{k \rightarrow \infty} (\pi + 2/j + 2/k) = \pi,$$

□

Example 3.7. Let (M^2, g_0) be the standard flat torus with S as in Example 3.6. Let $W_j = T_{1/j}(S)$ with respect to the flat norm. Let g_j be the flat metric on $M^2 \setminus W_j$. There exists smooth metrics g_j on M^2 which agree with g_0 on $M^2 \setminus W_j$ such that the Gromov-Hausdorff and Intrinsic Flat limits are the metric space created by taking the flat torus and identifying all points in S with each other. Then, g_j converges smoothly away from S to $g_\infty = g_0$. The metric completion of $(M \setminus S, g_\infty)$ is the slit torus as described in example 3.6. These metrics demonstrate that the diameter condition may not be replaced by an extrinsic diameter condition in Theorem 1.4 and in Theorem ?? but not the Ricci theorem since they have negative curvature.

Proof. Let $g_j = dt^2 + f_j(s, t)^2 ds^2$ where $f_j(s, t) = 1$ on W_j and $f_j(s, t) = 1/j$ on S , and smooth with values in $[1/j, 1]$ everywhere. Let \sim be defined as follows:

$$(90) \quad x \sim y \text{ iff } x, y \in S$$

To estimate the GH and SWIF distance between (M^2, g_j) and $(\frac{M^2}{\sim}, d_0)$ we use the Theorem 2.3. First we need to find an estimate on the distortion λ_j , which is defined by

$$(91) \quad \lambda_j = \sup_{x,y \in W_j} |d_{M_j}(x, y) - d_{\frac{M}{\sim}}(x, y)|.$$

Now let $P : M^2 \rightarrow \frac{M^2}{\sim}$ be the quotient map and Suppose $x_j, y_j \in \bar{W}_j$ achieve the maximum in the definition of λ_j . Since $\frac{M^2}{\sim}$ is flat outside $\frac{S}{\sim}$, any shortest path, \bar{C}_{x_j, y_j} , joining x_j, y_j has to be a straight line. As a result, $P^{-1}(\bar{C}_{x_j, y_j})$ is either the

straight line C_{x_j, y_j} , in M^2 joining x_j, y_j or the same straight line union the singular set S . And since the metric in (M^2, g_j) is smaller than the flat metric on outside W_j and coincide with the flat metric in W_j , we get

$$(92) \quad \lambda_j \leq L(\bar{C}_{x_j, y_j}) - L(C_{x_j, y_j})$$

$$(93) \quad \leq \text{diam}_{(M^2, g_j)}(M^2 \setminus W_j) + \text{diam}_{\left(\frac{M^2}{\sim}, d\right)}\left(\frac{M^2 \setminus W_j}{\sim}\right).$$

Any two points in $M^2 \setminus W_j$ can be joined by a few horizontal segments, whose lengths add up to at most $2/j$ and a segment in S with length less than π/j and vertical segments, whose lengths add up to $2/j$ therefore,

$$(94) \quad \text{diam}_{(M^2, g_j)}(M^2 \setminus W_j) \leq \frac{\pi + 4}{j},$$

and projecting these segments by P we get

$$(95) \quad \text{diam}_{\left(\frac{M^2}{\sim}, d\right)}\left(\frac{M^2 \setminus W_j}{\sim}\right) \leq 4/j,$$

hence,

$$(96) \quad \lambda_j \leq \frac{\pi + 8}{j} \rightarrow 0 \text{ as } j \rightarrow \infty.$$

Now letting $\epsilon = 0$ in Theorem 2.3, we have $a = 0$ and

$$(97) \quad \bar{h}_j = h_j = \sqrt{\lambda_j \left(\max \left\{ \text{diam}(W_j), \text{diam}\left(\frac{W_j}{\sim}\right) \right\} + \lambda_j/4 \right)} \rightarrow 0 \text{ as } j \rightarrow \infty.$$

So we conclude that

$$(98) \quad d_{GH} \left((M^2, g_j), \left(\frac{M^2}{\sim}, d \right) \right) \leq a + 2\bar{h} + \max \left\{ d_H^{M^2}(W_j, M^2), d_H^{\frac{M^2}{\sim}} \left(\frac{W_j}{\sim}, \frac{M^2}{\sim} \right) \right\} \rightarrow 0,$$

as $j \rightarrow \infty$ and also, it is easy to see that

$$(99) \quad \begin{aligned} d_{\mathcal{F}} \left((M^2, g_j), \left(\frac{M^2}{\sim}, d \right) \right) &\leq (\bar{h} + a) \left(\text{Vol}_2(W_j) + \text{Vol}_2\left(\frac{W_j}{\sim}\right) + \text{Vol}_1(\partial W_j) + \text{Vol}_1\left(\frac{\partial W_j}{\sim}\right) \right) \\ &\quad + \text{Vol}_2(M^2 \setminus W_j) + \text{Vol}_2\left(\frac{M^2}{\sim} \setminus \frac{W_j}{\sim}\right) \rightarrow 0 \text{ as } j \rightarrow \infty. \end{aligned}$$

As we observed,

$$(100) \quad \lim_{j \rightarrow \infty} \text{diam}_{(M^2, g_j)}(\partial W_j) \leq \lim_{j \rightarrow \infty} \frac{\pi + 4}{j} = 0,$$

but,

$$(101) \quad \lim_{j \rightarrow \infty} \text{diam}_{(W_j, g_j)}(\partial W_j) \geq \pi.$$

□

3.5. Splines with Positive Scalar Curvature. In this section, we will present two examples that demonstrate that in our Theorems, the uniform lower Ricci curvature bound condition can not be replaced by a uniform scalar curvature bound. In the first example we construct a sequence of metrics on the 3 - sphere which converge to the canonical sphere away from a singular point and also in the intrinsic flat sense but converges to a sphere with an interval attached to it in the Gromov-hausdorff sense. In the second example of this section, we will construct a sequence of metrics with positive scalar curvature which converge to the 3 - sphere away from a singular point and also in the intrinsic flat sense while having no Gromov-hausdorff limit. The second example was in fact presented by Tom Ilmannen in a talk in 2004 at Columbia without details. Both examples play an important role in [SW11] however the fact that they have positive scalar curvature was never presented in detail in that paper.

Lemma 3.8. *For any $L > 0$ and $0 < \delta < 1$, there exists a smooth Riemannian metric on the 3-sphere with positive scalar curvature which is obtained by properly gluing a spline of length $L + O(\delta^{\frac{1}{2}})$ and width $\leq \delta$ to the unit 3 - sphere.*

Example 3.9. *There are metrics g_j on the sphere M^3 with positive scalar curvature such that $M_j = (S^3, g_j)$ converge smoothly away from a point singularity $S = \{p_0\}$ to the sphere, S^3 , with $\text{diam}(M_j) \leq \pi + L + 2$ and such that*

$$(102) \quad d_{\mathcal{F}}(M_j, S^3), \quad d_{s\mathcal{F}}(M_j, S^3) \rightarrow 0,$$

and,

$$(103) \quad d_{GH}(M_j, M_0) \rightarrow 0,$$

where $M_0 = S^3 \sqcup [0, L]$ (the round sphere with an interval of length L attached to it).

Remark 3.10. *Example 3.9 demonstrates that the uniform lower Ricci curvature bound condition in Theorem 1.3 and 1.5 can not be replaced by a uniform lower bound on the scalar curvature.*

Example 3.11. *There are metrics g_j on the sphere M^3 with positive scalar curvature such that $M_j = (S^3, g_j)$ converge smoothly away from a point singularity $S = \{p_0\}$ to the sphere, S^3 , with $\text{diam}(M_j) \leq \pi + L + 2$ and such that*

$$(104) \quad d_{\mathcal{F}}(M_j, S^3), \quad d_{s\mathcal{F}}(M_j, S^3) \rightarrow 0,$$

and there is no Gromov-Hausdorff limit.

Proof. of Lemma 3.8. The goal here is to attach a spline of finite length and arbitrary small width to a sphere with positive scalar curvature. For this, we need to employ some ideas related to the Mass of rotationally symmetric manifolds. (c.f. [LS11]). The construction goes as follows; we first find an admissible Hawking mass function (c.f. [LS11]) which will provide us with a three manifold embedded in \mathbb{E}^4 which is a hemisphere to which spline of finite length and small width is attached; we then, attach a hemisphere along its boundary.

Let $\delta < 1$ (this later will become the width of the spline). and let $r_{min} = 0$. Now we take an admissible Hawking mass function, $m_H(r)$ (which has to be smooth and increasing) that satisfies (ϵ to be determined later)

$$(105) \quad m_H(r) = r(1 - \epsilon^2)/2 \text{ for } r \in [0, \delta^3],$$

and,

$$(106) \quad m_H(r) = r^3/2 \text{ for } r \in [\delta, 1].$$

As in [LS11], define the function $z(r)$ via

$$(107) \quad z(\bar{r}) = \int_{r_{min}}^{\bar{r}} \sqrt{\frac{2m_H(r)}{r - 2m_H(r)}} dr.$$

Note that z depend on δ .

$z(r)$ is unique up to a constant and gives our desired three manifold as a graph over \mathbb{E}^3 . By our choice of $m_H(r)$ we get,

$$(108) \quad z'(r) = \sqrt{\frac{1 - \epsilon^2}{\epsilon^2}} \text{ for } r \in [0, \delta^3],$$

and,

$$(109) \quad z'(r) = \sqrt{\frac{r^2}{1 - r^2}} \text{ for } r \in [\delta, 1],$$

and, since

$$(110) \quad \frac{\delta^3}{2}(1 - \epsilon^2) \leq m_H(r) \leq \delta^3/2 \text{ for } r \in [\delta^3, \delta],$$

one obtains

$$(111) \quad \sqrt{\frac{\delta^3(1 - \epsilon^2)}{r - \delta^3(1 - \epsilon^2)}} \leq z'(r) \leq \sqrt{\frac{\delta^3}{r - \delta^3}} \text{ for } r \in [\delta^3, \delta].$$

Now, choose the ϵ that solves

$$(112) \quad \delta^3 \sqrt{\frac{1 - \epsilon^2}{\epsilon^2}} = L,$$

For some fixed L . From (108), we have,

$$(113) \quad \bar{L}(\delta) = z(\delta) - z(\delta^3) \leq \int_{\delta^3}^{\delta} \sqrt{\frac{\delta^3}{r - \delta^3}} = 2\delta^{3/2}(\delta - \delta^3)^{1/2} < 2,$$

which goes to 0 as δ goes to 0.

We also get

$$(114) \quad z(\delta^3) - z(0) = \delta^3 \sqrt{\frac{1 - \epsilon^2}{\epsilon^2}} = L.$$

The metric in terms of the distance from the pole, can be written as

$$(115) \quad \bar{g}_\delta = ds^2 + f^2(s)g_{S^2} = (1 + [z'(r)]^2)dr^2 + r^2g_{S^2}.$$

In the virtue of the Theorem 5.4 in [LS11], we know that when $r \in [\delta, 1]$, we are on a unit sphere, and since

$$(116) \quad \lim_{r \rightarrow 1^-} z'(r) = \infty,$$

we have

$$(117) \quad \lim_{r \rightarrow 1^-} f'(s) = \lim_{r \rightarrow 1^-} r'(s) = \lim_{r \rightarrow 1^-} \frac{1}{\sqrt{1 + [z'(r)]^2}} = 0.$$

Therefore, the boundary $r = 1$ is in fact a great 2-sphere along which we can smoothly attach a 3 - hemisphere. as follows

So far we have got the metric

$$(118) \quad \bar{g}_\delta = (1 + [z'(r)]^2)dr^2 + r^2 g_{S^2} \text{ for } r \in [0, 1].$$

Letting $r = \sin(\rho)$, one sees that

$$(119) \quad \bar{g}_\delta = (1 + [z'_\delta(\sin(\rho))]^2) \cos^2(\rho) d\rho^2 + \sin^2(\rho) g_{S^2} \text{ for } \rho \in [0, \pi/2].$$

Therefore, on the sphere we define g_δ to be

$$(120) \quad (1 + [z'_\delta(\sin(\rho))]^2) \cos^2(\rho) d\rho^2 + \sin^2(\rho) g_{S^2} \text{ for } \rho \in [0, \pi/2],$$

and,

$$(121) \quad d\rho^2 + \sin^2(\rho) g_{S^2} \text{ for } \rho \in [\pi/2, \pi],$$

which has positive scalar curvature when $\rho \leq \pi/2$ because it is isometric to \bar{g}_δ and has positive scalar curvature when $\rho \geq \pi/2$ because it is isometric to a round hemisphere. g_δ is smooth at $\rho = \pi/2$ because by 109 near $\rho = \pi/2$,

$$(122) \quad z'_\delta(\sin(\rho)) = \sqrt{\frac{\sin^2(\rho)}{1 - \sin^2(\rho)}} = \tan(\rho).$$

So,

$$(123) \quad \begin{aligned} (1 + [z'_\delta(\sin(\rho))]^2) \cos^2(\rho) d\rho^2 + \sin^2(\rho) g_{S^2} &= (1 + \tan^2(\rho)) \cos^2(\rho) + \sin^2(\rho) g_{S^2} \\ &= d\rho^2 + \sin^2(\rho) g_{S^2}. \end{aligned}$$

The key idea is that, using this method, one can attach symmetric spline of length $L + \bar{L}(\delta)$ and arbitrary small width $\delta < 1$ to a sphere while keeping the scalar curvature positive and $\text{diam}(M_j) \leq \pi + L + 2$. And the metric found can actually be written as a warped metric.

□

Proof. of Example 3.9. Now let $\delta_j \rightarrow 0$ and take the sequence $M_j = (S^3, g_{\delta_j})$, where g_{δ_j} is given by the above construction for δ_j . we are going to prove that M_j converges to M_0 in Gromov-Hausdorff sense where M_0 is the unit three sphere to which an interval of length L is attached; and M_j converges to S^3 in intrinsic flat sense.

First notice that M_j contains a subdomain U_j which is isometric to $U'_j = S^3 \setminus B_p(\arcsin(\delta_j))$ also letting $V_j = M_j \setminus U_j$ and $V'_j = S^3 \setminus U'_j$ one observes that since

$$\begin{aligned}
 \text{Vol}(V_j) &\leq \int_0^\delta (4\pi r^2)(1 + [z'(r)]^2)^{1/2} dr \\
 (124) \quad &\leq \int_0^\delta (4\pi r^2)(1 + |z'(r)|)dr \\
 &\leq (4\pi\delta^2)(\delta + L + \bar{L}(\delta)),
 \end{aligned}$$

one gets $\text{Vol}(V_j) \rightarrow 0$ as $\delta_j \rightarrow 0$. Also it is obvious that $\text{Vol}(V'_j) \rightarrow 0$ as $\delta_j \rightarrow 0$.

Now, to be able to use Theorem 2.3, we need an estimate on

$$(125) \quad \lambda_j = \sup_{x,y \in U_j} |d_{M_j}(x,y) - d_{S^3}(x,y)|.$$

Let $x, y \in U_j$ and let γ and $c_{x,y}$ be the minimizing geodesic connecting x and y in (M_j, g_j) and S^3 (resp.). If γ lies completely in U_j , then, so does $c_{x,y}$ and $\gamma = c_{x,y}$ hence, $d_{M_j}(x,y) = d_{S^3}(x,y)$. If $\gamma \not\subset U_j$, therefore $\gamma = \gamma_1 + \gamma_2 + \gamma_3$ where $x \in \gamma_1$, $y \in \gamma_3$ and $\gamma_1, \gamma_3 \subset U_j$ and $\gamma_2 \subset V_j$. We are in either of the following cases

Case I: $c_{x,y} \subset U_j$

Obviously $L(\gamma_2) \leq 2\pi\delta_j$, also we have

$$(126) \quad |d_{S^3}(x, p) - L(\gamma_1)| \leq \arcsin \delta_j,$$

and

$$(127) \quad |d_{S^3}(y, p) - L(\gamma_3)| \leq \arcsin \delta_j.$$

Since $\delta_j \rightarrow 0$, for j large enough,

$$(128) \quad L(\gamma) \approx d_{S^3}(x, p) + d_{S^3}(y, p) > L(c_{x,y}),$$

which is a contradiction.

Case II: $c_{x,y} \not\subset U_j$.

Let $c_{x,y} = c_1 + c_2 + c_3$ where $x \in c_1$, $y \in c_3$ and $c_1, c_3 \subset U_j$ and $c_2 \subset V'_j$. Then, $L(c_2) \leq 2 \arcsin(\delta_j)$ and also

$$(129) \quad |L(\gamma_i) - L(c_i)| \leq 2 \arcsin(\delta_j).$$

Therefore,

$$(130) \quad |d_{M_j}(x,y) - d_{S^3}(x,y)| = |L(\gamma) - L(c_{x,y})| \leq 4 \arcsin(\delta_j) + 2\pi\delta_j.$$

This argument shows that $\lambda_j \rightarrow 0$ as $j \rightarrow \infty$. Since the intrinsic diameter $D_{U_j} \leq \pi$ (both in M_j and S^3), h_j in Theorem 2.3 goes to 0 as $j \rightarrow \infty$. Letting $\epsilon = 0$ in Theorem 2.3 we get $a = 0$ and $\bar{h}_j = h_j$ therefore,

$$(131) \quad d_{\mathcal{F}}(M_j, S^3) \leq \bar{h}_j (2 \text{Vol}(U_j) + 2 \text{Vol}(\partial U_j)) + \text{Vol}(V_j) + \text{Vol}(V'_j).$$

which gives

$$(132) \quad d_{\mathcal{F}}(M_j, S^3) \rightarrow 0 \text{ as } j \rightarrow \infty.$$

To prove that M_j converges to M_0 in Gromov-hausdorff sense, we will estimate $d_{GH}(M_j, M_0)$ using the fact that

$$(133) \quad d_{GH}(M_j, M_0) = \frac{1}{2} \inf_{\mathfrak{R}} (\text{dis } \mathfrak{R}),$$

where, the infimum is over all correspondences \mathfrak{R} between M_j and M_0 and $\text{dis } \mathfrak{R}$ is the distortion of \mathfrak{R} given by

$$(134) \quad \text{dis } \mathfrak{R} = \sup \{ |d_{M_j}(x, x')| - |d_{M_0}(y, y')| : (x, y), (x', y') \in \mathfrak{R} \}.$$

For details see [BBI01, p. 257].

We need to find correspondences \mathfrak{R}_j between M_j and M_0 such that $\text{dis } \mathfrak{R}_j \rightarrow 0$ as $j \rightarrow \infty$. Consider $W_j \subset \mathbb{E}^4$ given by

$$(135) \quad W_j = M_j \cup M_0 = M_0 \cup V_j = M_j \cup (M_0 \setminus U_j).$$

In fact, we can picture W_j as the union of sphere, an interval of length L and a spline of length $L + \bar{L}(\delta_j)$ around the spline. Then we have, $M_j \subset W_j$ and $M_0 \subset W_j$ define the following surjective maps $f : W_j \rightarrow M_j$ and $g : W_j \rightarrow M_0$

$$(136) \quad f|_{M_j} = \text{id}.$$

$$(137) \quad f(w) = (r(z(w)), 0, 0, z(w)) \in V_j \text{ for } w \in (M_0 \setminus U_j).$$

Note that when $w \in (M_0 \setminus U_j)$, $f(w)$ is the point in $V_j \cap xz$ -plane closest to w . Similarly let

$$(138) \quad g|_{M_0} = \text{id}.$$

$$(139) \quad g(w) = \text{the point in } (M_0 \setminus U_j) \text{ closest to } w \text{ for } w \in V_j.$$

Let \mathfrak{R}_j be the following correspondence between M_j and M_0 ,

$$(140) \quad \mathfrak{R}_j = \{(f(w), g(w)) : w \in W_j\}.$$

Claim: $\text{dis } \mathfrak{R}_j \rightarrow 0$ as $j \rightarrow \infty$.

Pick $w_1, w_2 \in W_j$, and suppose $\gamma + \lambda$ is the minimal geodesic in M_j connecting $f(w_1)$ and $f(w_2)$ where $\gamma \subset U_j$ and $\lambda \subset V_j$. and let $\gamma' + \lambda'$ be the (possibly) broken minimal geodesic connecting $g(w_1)$ and $g(w_2)$ where, $\gamma' \subset U_j$ and $\lambda' \subset (M_0 \setminus U_j)$. Without loss of generality we assume that $z(w_1) \leq z(w_2)$. Next we need estimates on the lengths of $\gamma, \lambda, \gamma', \lambda'$. Let q and q' be starting points on λ and λ' respectively, then

$$(141) \quad \int_q^{f(w_2)} dz \leq \int_q^{f(w_2)} s'(z) dz \leq L(\lambda) \leq \int_q^{f(w_2)} s'(z) dz + 2\pi\delta_j,$$

where, the term $2\pi\delta_j$ is the maximum perimeter of the well and note that any two point on the we can be joined by a radial geodesic followed by a curve of length less than $2\pi\delta_j$.

Since $ds^2 = (1 + [r'(z)]^2) dz^2$ we get

$$(142) \quad \int_q^{f(w_2)} s'(z) dz \leq \int_q^{f(w_2)} dz + \int_q^{f(w_2)} |r'(z)| dz \\ \leq \int_q^{f(w_2)} dz + \delta_j (L + \bar{L}(\delta_j)).$$

We also have

$$(143) \quad \int_{q'}^{g(w_2)} dz \leq L(\lambda') \leq \int_{q'}^{g(w_2)} dz + 2 \arcsin(\delta_j).$$

The last inequality comes from the fact that any two points in $M_0 \setminus U_j$ can be joined by a (broken) geodesic which is a straight line followed by a curve of length at most $\text{diam}(V'_j) = 2 \arcsin(\delta_j)$.

On the other hand by our construction

$$(144) \quad \left| \int_q^{q'} dz \right| \leq \bar{L}(\delta_j),$$

and,

$$(145) \quad \left| \int_{f(w_2)}^{g(w_2)} dz \right| \leq \bar{L}(\delta_j).$$

Therefore,

$$(146) \quad \left| \int_q^{f(w_2)} dz - \int_{q'}^{g(w_2)} dz \right| \leq 2\bar{L}(\delta_j).$$

From 141 - 146, we get

$$(147) \quad |L(\lambda) - L(\lambda')| \leq \delta_j (L + \bar{L}(\delta_j) + 2\pi) + 2\bar{L}(\delta_j) + 2 \arcsin(\delta_j).$$

Also one observes that when $w_1 \in U_j$, then γ and γ' are geodesics on the sphere starting from the same point and ending up in V'_j which means that

$$(148) \quad |L(\gamma) - L(\gamma')| \leq \text{diam}(V'_j) = 2 \arcsin(\delta_j).$$

From (147) and (148),

$$(149) \quad |d_{M_j}(f(w_1), f(w_2)) - d_{M_0}(g(w_1), g(w_2))| \leq |L(\gamma) - L(\gamma')| + |L(\lambda) - L(\lambda')| \\ \leq \delta_j (L + \bar{L}(\delta_j) + 2\pi) + 2\bar{L}(\delta_j) + 4 \arcsin(\delta_j).$$

Therefore,

$$(150) \quad \text{dis } \mathfrak{R}_j \leq \delta_j (L + \bar{L}(\delta_j) + 2\pi) + 2\bar{L}(\delta_j) + 4 \arcsin(\delta_j),$$

which shows that $\text{dis } \mathfrak{R}_j \rightarrow 0$ as $j \rightarrow \infty$. This completes the proof of the claim.

To prove that the convergence off the singular set $S = \{p_0\}$, which is the bottom of the well, let $\rho_0 > 0$, then

$$(151) \quad g_{\delta_j} \rightarrow g_{S^3} \text{ on } \rho^{-1}([\rho_0, \pi]),$$

because for j sufficiently large, $\delta_j < \rho_0$, which by our construction means that $g_{\delta_j} = g_{S^3}$ on $\rho^{-1}([\rho_0, \pi])$. \square

Proof. of Example 3.11. Let $g(p_0, s)$ denote a symmetric spline of length L centered at the point p_0 with width s (as constructed in the previous examples). Also fix a great circle and a point 0 in S^2 , so now, when we say a point given by the angle θ , it means a point on this great circle given by the angle θ . On the round sphere, $r = \pi - \frac{1}{2j}$ is a 2-sphere with radius $\sin(\pi - \frac{1}{2j})$. Therefore, balls with radius $s_j = \frac{1}{j} \sin(\pi - \frac{1}{2j}) \sqrt{2 - 2\cos(\frac{2\pi}{2j})}$ centered at points p_k given by the angle $\theta = \frac{2k\pi}{2j}$ are disjoint. Now for each j , we can glue metrics $g(p_k, s_j)$ which agree with the metric on the spline given in Example 3.9. Outside of each $B_{p_k}(s_j)$, we set $g_j = g_0$. It is easy to see that by our construction, g_j agrees with the round metric for $r < \pi - \frac{1}{2j} - s_j$ and has 2^j splines of length L and width s_j and also the volume of the non spherical part is going to 0 as $j \rightarrow \infty$. So by taking $U_j = r^{-1}([0, \pi - \frac{1}{2j} - s_j])$, we see that again all conditions in Theorem 2.10 are satisfied therefore we have the flat convergence to the settled completion. \square

4. HAUSDORFF MEASURE ESTIMATES \implies WELL-EMBEDDEDNESS.

We now prove Theorems 1.2, 1.3 and its counterpart (with Ricci condition replaced by contractibility condition) stated in the introduction. First we must prove the following two lemmas:

Lemma 4.1. *Let M^n be compact Riemannian manifold, S a subset of M with $H^{n-1}(S) = 0$, and let $\gamma : [0, L] \rightarrow M$ be a shortest geodesic parametrized by arclength with endpoints $x, y \in M \setminus S$. Then, for any small enough $\epsilon > 0$, there exists a path γ_ϵ joining x, y such that $\gamma_\epsilon \cap S = \emptyset$ and*

$$(152) \quad L(\gamma_\epsilon) \leq L(\gamma) + \epsilon.$$

Proof. Let $\Gamma : [-\sigma, \sigma]^{n-1} \times [0, L] \subset \mathbb{R}^n \rightarrow M$ be the $(n-1)$ -th variation of γ given by

$$(153) \quad \Gamma(t_1, t_2, \dots, t_{n-1}, s) = \exp_{\gamma(s)} \mathbf{F}(t_1, t_2, \dots, t_{n-1}, s)$$

where

$$(154) \quad \mathbf{F}(t_1, t_2, \dots, t_{n-1}, s) = \sin\left(\frac{\pi s}{L}\right) \left(\sum_i t_i e_i(s) \right),$$

in which $\{e_i(s)\}$ is a parallel orthonormal frame along γ and $e_0(s)$ is the unit tangent to γ .

For any $\bar{t} = (t_1, \dots, t_{n-1})$, the curve $\gamma_{\bar{t}}(s) := \Gamma(t_1, t_2, \dots, t_{n-1}, s)$ is a curve from x to y . If we choose σ sufficiently small then,

$$(155) \quad L(\gamma) \leq L(\gamma_{\bar{t}}) \leq L(\gamma) + \epsilon,$$

therefore, to prove the lemma, we need to find a \bar{t} such that

$$(156) \quad \gamma_{\bar{t}} \cap S = \emptyset.$$

Claim: There exists some $\sigma > 0$ such that after restricting the domain of Γ accordingly, for any small $\delta > 0$, Γ is bi-Lipschitz on

$$(157) \quad \Lambda_\delta = [-\sigma, \sigma]^{n-1} \times [\delta, L - \delta].$$

To see this, we need to compute the derivative of Γ . Let

$$(158) \quad x(u, s, t_1, \dots, t_{n-1}) = \exp_{\gamma(s)}(u\mathbf{F}(s, t_1, \dots, t_{n-1})),$$

for fixed s, t_1, \dots, t_{n-1} , as u ranges from 0 to 1, the curve $x(u, s, t_1, \dots, t_{n-1})$ is a geodesic segment from $\gamma(s)$ to $\Gamma(s, t_1, \dots, t_{n-1})$. As s varies, x is a variation through geodesics therefore,

$$(159) \quad D(\Gamma)\left(\frac{\partial}{\partial s}\right)(s, t_1, \dots, t_{n-1}) = \mathbf{J}(1),$$

where, \mathbf{J} is the Jacobi field along this geodesic segment, with the initial conditions $\mathbf{J}(0) = e_0(s)$, $\nabla_{\frac{\partial x}{\partial u}} \mathbf{J}(0) = \frac{\pi}{L} \cos\left(\frac{\pi s}{L}\right) \sum_i t_i e_i(s)$ since,

$$(160) \quad \begin{aligned} \nabla_{\frac{\partial x}{\partial u}} \mathbf{J}(0) &= \nabla_{\frac{\partial x}{\partial u}} \frac{\partial x}{\partial s}(0, s, t_1, \dots, t_{n-1}) \\ &= \nabla_{\frac{\partial x}{\partial s}} \frac{\partial x}{\partial u}(0, s, t_1, \dots, t_{n-1}) \\ &= \nabla_{\frac{\partial x}{\partial s}} \mathbf{F}(0, s, t_1, \dots, t_{n-1}) \\ &= \frac{\pi}{L} \cos\left(\frac{\pi s}{L}\right) \sum_i t_i e_i(s). \end{aligned}$$

Also for any $1 \leq i \leq n-1$ we have:

$$(161) \quad D(\Gamma)\left(\frac{\partial}{\partial t_i}\right)(s, t_1, \dots, t_{n-1}) = \left(\exp_{\gamma(s)}\right)_* \big|_{\mathbf{F}(s, t_1, \dots, t_{n-1})} \sin\left(\frac{\pi s}{L}\right) e_i(s).$$

For $t_1 = \dots, t_{n-1} = 0$, and $V = \alpha_0 \frac{\partial}{\partial s} + \sum_i \alpha_i \frac{\partial}{\partial t_i}$ with $\|V\| = 1$ we compute:

$$(162) \quad \begin{aligned} \|D(\Gamma)V\| &= \left\| D(\Gamma)\left(\alpha_0 \frac{\partial}{\partial s} + \sum_i \alpha_i \frac{\partial}{\partial t_i}\right) \right\| \\ &= \left\| \alpha_0 e_0(s) + \sin\left(\frac{\pi s}{L}\right) \sum_i \alpha_i e_i(s) \right\|, \end{aligned}$$

Therefore, for any $\delta > 0$, there exist $c(\delta) > 0$ such that on γ ,

$$(163) \quad 0 < c(\delta) \leq \|D(\Gamma)V\| \leq 1.$$

By continuity, for a small enough σ , we will have

$$(164) \quad 0 < \frac{c(\delta)}{2} \leq \|D(\Gamma)V\| \leq 3/2,$$

on $[-\sigma, \sigma]^{n-1} \times [\delta, L - \delta]$.

Also by making σ smaller, we can assume that

$$(165) \quad \sigma < \frac{r_{focal}}{n},$$

in which, r_{focal} is the focal radius of the geodesic γ . This guarantees that Γ is injective on $[-\sigma, \sigma]^{n-1} \times [\delta, L - \delta]$ which is compact, therefore Γ is a homeomorphism with the derivative bounded away from zero on its domain. By applying the inverse function theorem, we deduce that Γ is a diffeomorphism on $[-\sigma, \sigma]^{n-1} \times [\delta, L - \delta]$ onto its image. As a result, Γ is bi-Lipschitz on Λ_δ .

It is rather straightforward to see that for a Lipschitz function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, any subset $A \subset \mathbb{R}^n$ and $0 \leq s < \infty$, we have

$$(166) \quad H^s(f(A)) \leq [\text{Lip } f]^s H^s(A)$$

(see [LY02, Theorem 3.1.2]). Therefore, bi-Lipschitz preimages of sets of 0 Hausdorff measure, have 0 Hausdorff measure. Since Γ is bi-Lipschitz on Λ_δ , we get:

$$(167) \quad H^{n-1}(\Gamma^{-1}(S) \cap \Lambda_\delta) = 0.$$

Now we can compute:

$$\begin{aligned} H^{n-1}(\Gamma^{-1}(S)) &\leq H^{n-1}\left(\bigcup_i (\Gamma^{-1}(S) \cap \Lambda_{1/i})\right) \\ &\quad + H^{n-1}(\Gamma^{-1}(S) \cap [-\sigma, \sigma]^{n-1} \times \{0\}) \\ &\quad + H^{n-1}(\Gamma^{-1}(S) \cap [-\sigma, \sigma]^{n-1} \times \{L\}) \\ &= 0. \end{aligned}$$

Since any orthogonal projection $\text{Pr} : \mathbb{R}^n \rightarrow \mathbb{R}^k$ is distance decreasing, we have $\text{Lip}(\text{Pr}) \leq 1$. By (166) (see [LY02, Theorem 3.1.2]), for any $A \subset \mathbb{R}^n$ and any $0 \leq s < \infty$, we get

$$(168) \quad H^s(\text{Pr}(A)) \leq H^s(A)$$

Thus for any orthogonal projection Pr onto an $(n-1)$ -dimensional face of $[-\sigma, \sigma]^{n-1} \times [0, L]$ we have

$$(169) \quad H^{n-1}(\text{Pr}(\Gamma^{-1}(S))) = 0.$$

Let Pr_1 and Pr_2 be the projections onto the faces $[-\sigma, \sigma]^{n-1} \times \{0\}$ and $[-\sigma, \sigma]^{n-1} \times \{L\}$ respectively. Setting $E_1 = \text{Pr}_1(\Gamma^{-1}(S))$ and $E_2 = \text{Pr}_2(\Gamma^{-1}(S))$, then,

$$(170) \quad H^{n-1}(E_i) = 0 \text{ for } i = 1, 2$$

and so,

$$(171) \quad H^{n-1}([-\sigma, \sigma]^{n-1} \setminus E_i) = (2\sigma)^{n-1} \text{ for } i = 1, 2$$

Any countable union of null sets is a null set (see [Fol99, p. 26]) hence, $E_1 \cup E_2$ is a null set. This means that

$$(172) \quad H^{n-1}([-\sigma, \sigma]^{n-1} \setminus E_1 \cup [-\sigma, \sigma]^{n-1} \setminus E_2) = (2\sigma)^{n-1}.$$

Let $\bar{t} = (t_1, t_2, \dots, t_{n-1}) \in (([-\sigma, \sigma]^{n-1} \setminus E_0) \cap ([-\sigma, \sigma]^{n-1} \setminus E_L))$, then the path

$$(173) \quad \gamma_{\bar{t}}(s) = \Gamma(t_1, t_2, t_3, \dots, s),$$

is a path joining x, y and $\gamma_{\bar{t}} \cap S = \emptyset$ which also satisfies

$$(174) \quad |L(\gamma_\epsilon) - L(\gamma)| \leq \epsilon.$$

□

Lemma 4.2. *Let M^n be a compact Riemannian manifold, S a set with $H^{n-1}(S) = 0$ and $\text{diam}_{g_\infty}(M \setminus S) < \infty$ then, any connected precompact exhaustion, W_j , of $M^n \setminus S$ is uniformly well embedded.*

Proof. Suppose not.

Let $x_{i,j,k}, y_{i,j,k} \subset \bar{W}_j$ achieve to supremum in the definition of $\lambda_{i,j}$.

Since \bar{W}_j is compact, a subsequence as $k \rightarrow \infty$ converges to $x_{i,j}, y_{i,j} \subset \bar{W}_j$. Let $\gamma_{i,j}$ be a minimizing geodesic between these points in M with respect to g_i . Since S is a set of codimension strictly larger than 1, by applying Lemma 4.1, we can find a curve $C_{i,j} : [0, 1] \rightarrow M \setminus S$ between these points such that

$$(175) \quad L_{g_i}(C_{i,j}) \leq d_{M,g_i}(x_{i,j}, y_{i,j}) + \lambda_{i,j}/5,$$

Let k be chosen from the subsequence sufficiently large that

$$\begin{aligned} C_{i,j}([0, 1]) &\subset W_k, \\ d_{(\bar{W}_j, g_i)}(x_{i,j,k}, x_{i,j}) &< \lambda_{i,j}/10, \\ d_{(\bar{W}_j, g_i)}(y_{i,j,k}, y_{i,j}) &< \lambda_{i,j}/10, \end{aligned}$$

Thus

$$\begin{aligned} d_{(\bar{W}_k, g_i)}(x_{i,j,k}, y_{i,j,k}) &\leq d_{(\bar{W}_k, g_i)}(x_{i,j,k}, x_{i,j}) + d_{(\bar{W}_k, g_i)}(x_{i,j}, y_{i,j}) + d_{(\bar{W}_k, g_i)}(y_{i,j}, y_{i,j,k}) \\ &\leq d_{(\bar{W}_j, g_i)}(x_{i,j,k}, x_{i,j}) + L(C_{i,j}) + d_{(\bar{W}_j, g_i)}(y_{i,j}, y_{i,j,k}) \\ &\leq \lambda_{i,j}/10 + d_{M,g_i}(x_{i,j}, y_{i,j}) + \lambda_{i,j}/5 + \lambda_{i,j}/10 \\ &\leq 2\lambda_{i,j}/5 + d_{M,g_i}(x_{i,j}, x_{i,j,k}) + d_{M,g_i}(x_{i,j,k}, y_{i,j,k}) + d_{M,g_i}(y_{i,j,k}, y_{i,j}) \\ &\leq 2\lambda_{i,j}/5 + d_{W_j, g_i}(x_{i,j}, x_{i,j,k}) + d_{M,g_i}(x_{i,j,k}, y_{i,j,k}) + d_{W_j, g_i}(y_{i,j,k}, y_{i,j}) \\ &\leq 3\lambda_{i,j}/5 + d_{M,g_i}(x_{i,j,k}, y_{i,j,k}) \\ &\leq 3\lambda_{i,j}/5 + d_{W_k, g_i}(x_{i,j,k}, y_{i,j,k}) - \lambda_{i,j,k}, \end{aligned}$$

by the choice of $x_{i,j,k}$ and $y_{i,j,k}$. This is a contradiction.

Next we must show

$$(176) \quad \limsup_{j \rightarrow \infty} \limsup_{k \rightarrow \infty} \limsup_{i \rightarrow \infty} \lambda_{i,j,k} \leq \lambda_0.$$

Observe that

$$(177) \quad \lambda_{i,j,k} \leq \bar{\lambda}_{i,j,k} = \text{diam}_{(W_k, g_i)}(W_j).$$

Since $g_i \rightarrow g_\infty$ on W_k we know

$$(178) \quad \limsup_{i \rightarrow \infty} \lambda_{i,j,k} \leq \text{diam}_{(W_k, g_\infty)}(W_j).$$

Claim:

$$(179) \quad \limsup_{k \rightarrow \infty} \text{diam}_{(W_k, g_\infty)}(W_j) \leq \text{diam}_{(M \setminus S, g_\infty)}(W_j).$$

Suppose not; then, there exists $s > 0$ and a subsequence $k \rightarrow \infty$ such that

$$(180) \quad \limsup_{k \rightarrow \infty} \text{diam}_{(W_k, g_\infty)}(W_j) = L > \text{diam}_{(M \setminus S, g_\infty)}(W_j) + 5\delta.$$

Pick $x_k, y_k \in W_j$ so that

$$(181) \quad \text{diam}_{(W_k, g_\infty)}(W_j) \leq d_{(W_k, g_\infty)}(x_k, y_k) + \delta.$$

\bar{W}_j is compact therefore, after passing to a subsequence,

$$(182) \quad x_k \rightarrow x \in \bar{W}_j$$

$$(183) \quad y_k \rightarrow y \in \bar{W}_j.$$

Therefore, there exists a curve $c : [0, 1] \rightarrow M \setminus S$ such that,

$$(184) \quad L_{g_\infty}(c) < d_{(M \setminus S, g_\infty)}(x, y) + \delta$$

$$(185) \quad < \text{diam}_{(M \setminus S, g_\infty)}(\bar{W}_j) + \delta$$

$$(186) \quad \leq \text{diam}_{(M \setminus S, g_\infty)}(W_j) + \delta$$

$$(187) \quad < L - 4\delta.$$

For k sufficiently large, we have

$$(188) \quad c([0, 1]) \subset W_k,$$

so

$$(189) \quad L - 4\delta > L_{g_\infty}(c) > d_{(W_k, g_\infty)}(x, y)$$

$$(190) \quad > d_{(W_k, g_\infty)}(x_k, y_k) - 2\delta$$

$$(191) \quad \geq \text{diam}_{(W_k, g_\infty)}(W_j) - 3\delta.$$

Taking the limit as $k \rightarrow \infty$, we get:

$$(192) \quad L - 4\delta \geq L.$$

which is a contradiction hence, the claim is proved and we have

$$(193) \quad \limsup_{k \rightarrow \infty} \limsup_{i \rightarrow \infty} \lambda_{i,j,k} \leq \text{diam}_{(M \setminus S, g_\infty)}(W_j),$$

and so

$$(194) \quad \limsup_{j \rightarrow \infty} \limsup_{k \rightarrow \infty} \limsup_{i \rightarrow \infty} \lambda_{i,j,k} \leq \text{diam}_{g_\infty}(M \setminus S).$$

□

Proof of Theorem 1.2:

Proof. The lemmas 4.1 and 4.2 prove the well-embeddedness and then applying the Theorem 2.10 completes the proof of Theorem 1.2. □

Remark 4.3. *Example 3.5 demonstrates that the connectivity of the exhaustion in Theorems 1.2 hence, in Theorems 1.3 and 4.6 is a necessary condition. The excess volume bound in (23) is shown to be necessary in [LS, Example 3.7]. All these examples satisfy the uniform embeddedness hypothesis of Theorem 2.10 and demonstrate the necessity of these conditions in that theorem as well. By Lemma 2.14, the diameter hypothesis is not necessary when the Ricci curvature is nonnegative*

although the volume condition is still necessary as seen in [LS, Example 3.8]. Otherwise we see this is a necessary condition in Example 3.3. We were unable to find an example proving the necessity of the uniform bound on the boundary volumes, (22), and suggest this as an open question in [LS, Remark 3.15]. The Hausdorff measure condition $H^{n-1}(S) = 0$ of Theorem 1.2 and the uniform embeddedness hypothesis of Theorem 2.10 are seen to be necessary for their respective theorems in 3.6.

Proof of Theorem 1.3:

Proof. The assumption $H^{n-1}(S) = 0$ along with the hypotheses (50), (22) and (23), allows us to apply Theorem 1.2. Therefore, (M_i, g_i) has an intrinsic flat limit and this limit coincides with the settled completion of $(M \setminus S, g_\infty)$. Now by proposition 2.13, the Gromov-Hausdorff and Intrinsic Flat limits agree. \square

Remark 4.4. Example 3.3 proves the necessity of the condition (50) in Theorem 1.3.

Remark 4.5. From [CC00], Ricci bounded below and $d_{GH}(M_i, M) \rightarrow 0$ imply $\text{Vol}(M_i) \rightarrow \text{Vol}(M)$ (conjectured by Anderson-Cheeger) and M_i are homeomorphic to M for i sufficiently large (later proved diffeomorphic in [Per94] .) This means that (23) in Theorem 1.3 is a necessary condition.

Theorem 4.6. Let $M_i = (M, g_i)$ be a sequence of oriented compact Riemannian manifolds with a uniform linear contractibility function, ρ , which converges smoothly away from a closed singular subset, S , with $H^{n-1}(S) = 0$. If there is a connected precompact exhaustion of $M \setminus S$ as in (8) satisfying the volume conditions

$$(195) \quad \text{Vol}_{g_i}(\partial W_j) \leq A_0$$

and

$$(196) \quad \text{Vol}_{g_i}(M \setminus W_j) \leq V_j \text{ where } \lim_{j \rightarrow \infty} V_j = 0,$$

then

$$(197) \quad \lim_{j \rightarrow \infty} d_{GH}(M_j, N) = 0,$$

where N is the settled and metric completion of $(M \setminus S, g_\infty)$.

Proof. By the proof of Theorem 2.15, we see that

$$(198) \quad \text{diam}(M_i) \leq D_0,$$

This along with $H^{n-1}(S) = 0$, (195) and (196), allows us to apply the Lemma 4.2 to get the well-embeddedness of the exhaustion $\{W_j\}$. Then, we can fully apply Theorem 2.15 and that finishes the proof. \square

5. DIAMETER CONTROLS \implies WELL-EMBEDDEDNESS.

In this section we prove Theorems 1.4, 1.5 and its counterpart (with Ricci condition replaced with contractibility condition.) In the theorems of this section, there is no co-dimension condition on the singular set S . We first need to prove the following lemma:

Lemma 5.1. *Suppose W_j is a connected precompact exhaustion of $M \setminus S$ with boundaries ∂W_j such that any connected component of $M \setminus W_j$ has a connected boundary. If the intrinsic diameters satisfy*

$$(199) \quad \text{diam}_{(W_j, g_i)}(W_j) \leq D_{\text{int}},$$

and

$$(200) \quad \limsup_{i \rightarrow \infty} \left\{ \sum_{\beta} \text{diam}_{(\Omega_j^\beta, g_i)}(\Omega_j^\beta) : \Omega_j^\beta \text{ connected component of } \partial W_j \right\} \leq B_j$$

satisfies $\lim_{j \rightarrow \infty} B_j = 0$ then W_j is uniformly well embedded.

Proof. Recall from Definition 2.9 that we have

$$(201) \quad \lambda_{i,j,k} = \sup_{x,y \in W_j} |d_{(W_k, g_i)}(x, y) - d_{(M, g_i)}(x, y)| \leq \text{diam}_{(W_k, g_i)}(W_j).$$

Since

$$(202) \quad \text{diam}_{W_k, g_i}(W_j) \leq \text{diam}_{W_k, g_i}(W_k),$$

and g_i converges smoothly on W_k we have,

$$(203) \quad \lim_{i \rightarrow \infty} \text{diam}_{(W_k, g_i)}(W_k) = \text{diam}_{(W_k, g_\infty)}(W_k) \leq D_{\text{int}}.$$

Therefore,

$$(204) \quad \limsup_{j \rightarrow \infty} \limsup_{k \rightarrow \infty} \limsup_{i \rightarrow \infty} \lambda_{i,j,k} \leq \limsup_{j \rightarrow \infty} \limsup_{k \rightarrow \infty} \lim_{i \rightarrow \infty} \text{diam}_{(W_k, g_i)}(W_k) \leq D_{\text{int}}.$$

Now suppose $x_{ijk}, y_{ijk} \in \partial W_j$ give the supremum in the definition of λ_{ijk} . Let γ_{ijk} and C_{ijk} be shortest paths between x_{ijk} and y_{ijk} in \bar{W}_k and M respectively. Letting $k \rightarrow \infty$ and passing to a subsequence if necessary, $x_{ijk} \rightarrow x_{ij} \in \bar{W}_j$ and $y_{ijk} \rightarrow y_{ij} \in \bar{W}_j$. Passing to a subsequence again if necessary, C_{ijk} converges to C_{ij} which is a shortest path between x_{ij} and y_{ij} in M (c.f. [BBI01][Prop 2.5.17]). And let γ_{ij}^k be the shortest path between x_{ij} and y_{ij} in \bar{W}_k .

We will estimate C_{ij} by curves in ∂W_j with controlled increase in the length. Denote the curve obtained in n th step by C_{ij}^n and let $C_{ij}^0 = C_{ij}$. To obtain C_{ij}^{n+1} from C_{ij}^n , we proceed as follows: Suppose $\{\Omega_j^n\}_{n \in \mathbb{N}}$ are the connected components met by C_{ij}^n (in more than one point). If C_{ij}^n does not intersect Ω_j^{n+1} , then we let $C_{ij}^{n+1} = C_{ij}^n$. If C_{ij}^n intersects Ω_j^{n+1} then define

$$(205) \quad t_1 = \inf \{t : C_{ij}^n(t) \in \Omega_j^{n+1}\},$$

and

$$(206) \quad t_2 = \sup \{t : C_{ij}^n(t) \in \Omega_j^{n+1}\}.$$

Since Ω_j^{n+1} is connected, we can replace the segment $C_{ij}^n[t_1, t_2]$ with a shortest path in Ω_j^{n+1} . The curve obtained in this way is our C_{ij}^{n+1} . Note that connectivity of the boundary components of $M \setminus W_j$ implies that if C_{ij}^n enters $M \setminus W_j$ through Ω_j^{n+1} at time t , then it has to intersect Ω_j^{n+1} again at time $t' > t$ in order to enter W_j .

This construction implies that for all n ,

$$(207) \quad L(C_{ij}^n) \leq L(C_{ij}^0) + B_j,$$

hence, the sequence $\{C_{ij}^n\}_{n \in \mathbb{N}}$ obtained in this way have uniform bounded length and as a result, we can apply the Arzela-Ascoli's theorem to obtain, after possibly passing to a subsequence, a limit C'_{ij} i.e. C'_{ij} is a curve with end points x_{ij}, y_{ij} and there are parametrizations of $\{C_{ij}^n\}_{n \in \mathbb{N}}$ and C'_{ij} on the same domain such that $\{C_{ij}^n\}_{n \in \mathbb{N}}$ uniformly converges to C'_{ij} . We claim that C'_{ij} is contained in \bar{W}_j . For any t , tracing the curve C_{ij}^0 back and forth from the point $C_{ij}^0(t)$, we reach two immediate components Ω_j^l and Ω_j^m and this means that for $n \geq \max\{l, m\}$, $C_{ij}^n(t) \in \bar{W}_j$ and since $C_{ij}^n(t) \rightarrow C'_{ij}(t)$ we must have $C'_{ij}(t) \in \bar{W}_j$. Furthermore, for i large enough (depending on j)

$$(208) \quad \begin{aligned} \limsup_{i \rightarrow \infty} (L(\gamma_{ij}^k) - L(C_{ij})) &\leq \limsup_{i \rightarrow \infty} (L(C'_{ij}) - L(C_{ij})) \\ &\leq \limsup_{i \rightarrow \infty} \sum_{\beta} \text{diam}_{\Omega_j^\beta}^{g_i}(\Omega_j^\beta) \\ &< B_j. \end{aligned}$$

Also

$$(209) \quad \begin{aligned} \lim_{k \rightarrow \infty} \lambda_{ijk} &= \lim_{k \rightarrow \infty} (d_{(W_k, g_i)}(x_{ijk}, y_{ijk}) - d_{(M, g_i)}(x_{ijk}, y_{ijk})) \\ &\leq \lim_{k \rightarrow \infty} (d_{(W_m, g_i)}(x_{ijk}, y_{ijk}) - d_{(M, g_i)}(x_{ijk}, y_{ijk})) \\ &= L(\gamma_{ij}^m) - L(C_{ij}). \end{aligned}$$

Therefore, combining (208) and (209) we have

$$(210) \quad \lambda_j = \limsup_{i \rightarrow \infty} \lim_{k \rightarrow \infty} \lambda_{ijk} \leq B_j,$$

hence,

$$(211) \quad \limsup_{j \rightarrow \infty} \lambda_j \leq \lim_{j \rightarrow \infty} B_j = 0.$$

□

Proof of Theorem 1.4:

Proof. The Lemma 5.1 combined with Theorem 2.10 completes the proof of Theorem 1.4. Recall Definition 2.9. □

Remark 5.2. Example 3.5 demonstrates that the connectivity of the exhaustion in Theorem 1.4, hence in Theorems 1.5 and 5.5 is a necessary condition. Example 3.7 demonstrates that in Theorem 1.4, hence, in Theorems 1.5 and 5.5, the condition

on the intrinsic diameter can not be replaced by the same condition on the extrinsic diameter. The necessity of other conditions follow as in Remark 4.3.

Proof of Theorem 1.5:

Proof. The hypothesis $\text{diam}(M_j) \leq D_0$ combined with the hypothesis including (22) and (23), allows us to apply Theorem 1.4. So (M_i, g_i) has an intrinsic flat limit and this intrinsic flat limit is the settled completion of $(M \setminus S, g_\infty)$. Thus by Proposition 2.13, the Gromov-Hausdorff and Intrinsic Flat limits agree. \square

Remark 5.3. Example 3.3 proves the necessity of the condition (20) in Theorem 1.5.

Remark 5.4. From [CC00], Ricci bounded below and $d_{GH}(M_i, M) \rightarrow 0$ imply $\text{Vol}(M_i) \rightarrow \text{Vol}(M)$ (conjectured by Anderson-Cheeger) and M_i are homeomorphic to M for i sufficiently large (later proved diffeomorphic in [Per94] .) This means that (23) in Theorem 1.5 is a necessary condition.

Theorem 5.5. Let $M_i = (M, g_i)$ be a sequence of Riemannian manifolds with a uniform linear contractibility function, ρ , which converges smoothly away from a closed singular set, S , with

$$(212) \quad \text{Vol}(M_i) \leq V_0.$$

If there is a connected precompact exhaustion, W_j , of $M \setminus S$, satisfying (8) such that each connected component of $M \setminus W_j$ has a connected boundary, satisfying (19)-(21), (2) and (16) then

$$(213) \quad \lim_{j \rightarrow \infty} d_{GH}(M_j, N) = 0.$$

where N is the metric completion of $(M \setminus S, g_\infty)$.

Proof. By Lemma 2.12, we have

$$(214) \quad \text{Vol}(M_i) \leq V_0.$$

This combined with the uniform contractibility function allows us to apply the Greene-Petersen Compactness Theorem. In particular we have a uniform upper bound on diameter

$$(215) \quad \text{diam}(M_i) \leq D_0,$$

We may now apply Theorem 1.4 to obtain

$$(216) \quad \lim_{j \rightarrow \infty} d_{\mathcal{F}}(M_j, N') = 0.$$

We then apply Theorem 2.7 to see that the flat limit and Gromov-Hausdorff limits agree due to the existence of the uniform linear contractibility function and the fact that the volume is bounded below uniformly by the smooth limit. In particular the metric completion and the settled completion agree. \square

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